Math 308N **Midterm 2** **Winter 2018**

Your Name

Your Signature

Student ID #

Honor Statement

I agree to complete this exam without unauthorized assistance from any person, materials, or device.

Signature: ______________________________

- Turn off all cell phones, pagers, radios, mp3 players, and other similar devices.
- This exam is closed book. You may use one $8.5'' \times 11''$ sheet of handwritten notes (both sides OK). Do not share notes. No photocopied materials are allowed.
- Only the TI 30X IIS calculators is allowed.
- In order to receive credit, you must **show all of your work**. If you do not indicate the way in which you solved a problem, you may get little or no credit for it, even if your answer is correct.
- If you need more room, use the backs of the pages and indicate that you have done so.
- Raise your hand if you have a question.
- This exam has 7 pages, plus this cover sheet. Please make sure that your exam is complete.

Question	Points	Score
1	24	
2	8	
3	10	
4	10	
5	8	
Total	60	

1. (24 points) Indicate whether the given statement is true or false (1 pts) and give justification as to why it is true or false(2 pts).

a) [4 pts] If S is a subspace of $\mathbb{R}^8$ and $\{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_7\}$ is a basis for S, then for any $\vec{v} \notin S$, span$\{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_7, \vec{v}\} = \mathbb{R}^8$.

> TRUE. If $\{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_7\}$ is a basis for S, then $\{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_7\}$ is linearly independent. Since $\vec{v} \notin S$, $\vec{v}$ is not in the span of the $\vec{u}_i$, which means that $\{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_7, \vec{v}\}$ is a linearly independent set in $\mathbb{R}^8$, hence spans $\mathbb{R}^8$.

b) [4 pts] Let B_1, B_2, and B_3 be bases for $\mathbb{R}^n$. If C_1 is the change of basis matrix going from B_1 to B_2, and C_2 is the change of basis matrix going from B_2 to B_3, then $C_2^{-1}C_1^{-1}$ is the change of basis matrix going from B_3 to B_1.

> FALSE. If C_1 is the change of basis matrix going from B_1 to B_2, and C_2 is the change of basis matrix going from B_2 to B_3, then C_2C_1 is the change of basis matrix going from B_1 to B_3, hence it's inverse goes from B_3 to B_1. By shoes and socks, the inverse is given by $(C_2C_1)^{-1} = C_1^{-1}C_2^{-1}$.

c) [4 pts] If W is a subspace of $\mathbb{R}^9$, $dim(W) = 3$, and $T : \mathbb{R}^9 \to \mathbb{R}^6$ is a linear transformation such that Ker(T)=W, then T must be onto.

> TRUE. By the rank-nullity theorem, we know that dim(range(T))+dim(ker(T))=9. Since ker(T)=W and $dim(W) = 3$, dim(ker(T))$= 3$. This means that dim(range(T))$+3 = 9$ which implies that dim(range(T))$= 6$. Since the codomain is $\mathbb{R}^6$, dim(range(T))$= 6$ implies that T is onto.

Give an example of each of the following. If it is not possible write "NOT POSSIBLE", and **give justification as to why**.

g) [2 pt] An $n \times n$ matrix $A \neq I_2$ such that $A^{2018} = I_2$, but $A^k \neq I_2$ for all $k < 2018$..

Take a rotation matrix for $\theta = \frac{2\pi}{2018}$, this looks like

$$C_{\frac{2\pi}{2018}} = \begin{bmatrix} \cos(\frac{2\pi}{2018}) & -\sin(\frac{2\pi}{2018}) \\ \cos(\frac{2\pi}{2018}) & sin(\frac{2\pi}{2018}) \end{bmatrix}$$

Since rotating through an angle of $\frac{2\pi}{2018}$, 2018 times will get you back where you started, $(C_{\frac{2\pi}{2018}})^{2018} = I_2$. Note that one can also just say $A = \begin{bmatrix} x & 0 \\ 0 & x \end{bmatrix}$ where $x^{2018} = 1$.

h) [2 pt] A linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ such that range(T)=ker(T).

Recall from class that when you look at a 2×2 matrix, the columns are reading off the image of the standard basis vectors. By rank-nullity, dim(range(T))+dim(ker(T))= 2, so range(T) and ker(T) must both be 1-dimensional. Lets pick a subspace and make it work. Picking the x or y axis will be easiest so lets pick the subspace to be span$\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$, the x-axis. If I want the range AND kernel to be this subspace, I must send $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ to $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$, to get the kernel. Moreover, I must send the other vector to $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, to get the range. The matrix that does this is $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. To see it more explicitly, reaidng off from the columns of the matrix shows that e_1 will go to the zero vector, and e_2 will go to e_1 (because column 2 of the matrix is e_1. Now looking at any arbitrary vector, we can see that any vector of the form $\begin{bmatrix} a \\ 0 \end{bmatrix}$ will go to the zero vector, and any vector of the form $\begin{bmatrix} 0 \\ b \end{bmatrix}$ will go to $\begin{bmatrix} b \\ 0 \end{bmatrix}$. This means that anything on the x-axis goes to zero and anything on the y-axis goes to the x-axis, which is tha same as $ker(T) = range(T) = span\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Note: There are certainly other choices that work and just writing a matrix that works is sufficient for full credit.

i) [2 pts] A basis B for $\mathbb{R}^3$ such that every vector in the basis lies in the set $\left\{ \begin{bmatrix} a \\ b \\ c \end{bmatrix} : a + b + 2c = 0 \right\}$

NOT POSSIBLE. The set $\left\{ \begin{bmatrix} a \\ b \\ c \end{bmatrix} : a + b + 2c = 0 \right\}$ forms a plane in $\mathbb{R}^3$. If this set were a basis then it would imply that a plane spans $\mathbb{R}^2$ but a plane can never span all of $\mathbb{R}^3$.

2. (10 points) Consider the matrix A, and it's reduced echelon form below

$$A = \begin{bmatrix} 2 & -6 & 14 & 4 & 18 \\ -1 & 6 & -19 & 4 & -6 \\ -2 & 7 & -18 & 1 & -11 \\ 3 & -8 & 17 & 3 & 18 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -5 & 0 & -4 \\ 0 & 1 & -4 & 0 & -3 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

a) [3 pts] Find a basis for $\mathrm{Col}(A)$, the column space of A.

By recipe 2, the pivot columns in the reduced echelon form occur in columns $1, 2$ and 4, so looking back at matrix A, we have that

$$B_{col(A)} = \left\{ \begin{bmatrix} 2 \\ -1 \\ -2 \\ 3 \end{bmatrix}, \begin{bmatrix} -6 \\ 6 \\ 7 \\ -8 \end{bmatrix}, \begin{bmatrix} 4 \\ 4 \\ 1 \\ 3 \end{bmatrix} \right\}$$

b) [3 pts] Find a basis for $\mathrm{Null}(A)$, the Null space of A.

Looking at the linear system corresponding to the reduced echelon matrix, we see that the free variables correspond to x_3 and x_5, so let $x_3 = s_1$ and $x_5 = s_2$. The remaining equations are then

$$x_1 - 5s_1 - 4s_2 = 0; \quad x_2 - 4s_1 - 3s_2 = 0; \quad x_4 + 2s_2 = 0$$

This means that

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 5s_1 + 4s_2 \\ 4s_1 + 3s_2 \\ s_1 \\ -2s_2 \\ s_2 \end{bmatrix} = s_1 \begin{bmatrix} 5 \\ 4 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s_2 \begin{bmatrix} 4 \\ 3 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

so

$$B_{Null(A)} = \left\{ \begin{bmatrix} 5 \\ 4 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \\ 0 \\ -2 \\ 1 \end{bmatrix} \right\}$$

c) [2 pts] Find a basis for $\mathrm{row}(A^T)$, the row space of A^T.

Since the rows of A^T are just the columns of A, row(A^T) =col(A), hence the basis is the same as in part a).

3. (10 points) a) [5 pts] Consider the set $S = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : x^2 + y^2 + z^2 \leq 1 \right\}$ Determine if S is a subspace of $\mathbb{R}^3$. If it is, carefully show that it is a subspace. If it is not, give justification as to why.

Algebraic Solution: S is not a subspace because it isn't closed under scalar multiplication nor vector addition. The easiest way to see this is via scalar multiplication. Given $\vec{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in S$ take any scalar $c > 1$, then $(cx)^2 + (cy)^2 + (cz)^2 = c^2(x^2 + y^2 + z^2)$ and since $x^2 + y^2 + z^2 \leq 1$, $c^2(x^2 + y^2 + z^2) \leq c^2 \not\leq 1$. This means that $c\vec{v} \notin S$, so S is not a subspace. S is also not closed under vector addition, as can easily be seen by considering (among others) the vectors $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$. Both are in S but their vector sum is not.

Geometric Solution: S is a sphere of radius 1 in $\mathbb{R}^3$. Since any vector whose tip is on the boundary of the sphere can be scaled to strech outside of the sphere, a scaled version of any vector in S will not necessarily remain in S.

b) [5 pts] Let A be an non-zero $n \times n$ matrix and consider the set $S = \left\{ x \in \mathbb{R}^n : A\vec{x} = \vec{x} \right\}$. That is, for some fixed matrix A, S consists of all vectors in $\mathbb{R}^n$ that are fixed by A. Show that S is a subspace of $\mathbb{R}^n$. (Note: You may use the definition or any theorems from class.)

Solution 1: Recall that for any matrix A, Null(A) is automatically a subspace. Keeping this fact in mind, we can see that if $A\vec{x} = \vec{x}$, then $A\vec{x} - I_n\vec{x} = (A - I_n)\vec{x} = \vec{0}$. This means that $S =$Null$(A - I_n)$. Since S is the nullspace of some matrix, it is a subspace.

Solution 2: We check the 3 subspace conditions starting with the first. It's easy to see that $\vec{0} \in S$ because $A\vec{0} = \vec{0}$.
Now let $\vec{x} \in S$ and let c be a scalar. Then

$$Ac\vec{x} = c(A\vec{x}) = c(\vec{x}) = c\vec{x}$$

so $c\vec{x} \in S$ and S is closed under scalar multiplication.
Lastly, let $\vec{x}, \vec{y} \in S$. This means that $A\vec{x} = \vec{x}$ and $A\vec{y} = \vec{y}$, since A is linear, we have

$$A(\vec{x} + \vec{y}) = A\vec{x} + A\vec{y} = \vec{x} + \vec{y}$$

hence $\vec{x} + \vec{y} \in S$, and S is closed under vector addition.

c)[4pts] Given the set S from part b), find a basis of S for the matrix $A = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$

We will approach this problem from the nullspace perspective but mention that one can solve this by letting $\vec{x} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ be an arbitrary and writing out equations in terms of a, b, and c, then solving.

Since $S = Null(A - I_n)$, here we have $n = 3$ and

$$A - I_3 = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

The linear system this corresponds to is given by

$$-x_1 + x_2 + x_3 = 0 \quad \text{and} \quad x_1 = 0$$

This means that x_3 is free, hence $x_3 = s$ and $x_2 = -s$. Since $S =$Null$(A - I_3)$, a basis for S is just a basis for this null space. We can now see that any vector in this null space has the form $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = s \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$ hence

$$B_S = \left\{ \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \right\}$$

4. (10 points) a) [5 pts] Define linear transformations $T_A : \mathbb{R}^3 \to \mathbb{R}^3$, $S_B : \mathbb{R}^3 \to \mathbb{R}^3$, and $R = (T \circ S) : \mathbb{R}^3 \to \mathbb{R}^3$ with $T_A(\vec{x}) = A\vec{x}$ and $S_B(\vec{x}) = B\vec{x}$ for $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -3 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$.
Show that T and S are invertible. (Note: Even though matrix B is the same as in the previous problem, they are unrelated.)

T and S are invertible $\Leftrightarrow \det(A) \neq 0$ and $\det(B) \neq 0$. It's easy to see that $\det(A) = (2)(-1)(3) = -6$, and computing the cofactor expansion along the first column shows that $\det(B) = 1(-1) = -1$, hence T and S are invertible.

b) [5 pts] Determine a matrix C such that $R^{-1}(\vec{x}) = (T \circ S)^{-1}(\vec{x}) = C\vec{x}$.

Since matrix multiplication and function composition are the same, $(T \circ S)\vec{x} = AB\vec{x}$. This means that $(T \circ S)^{-1}$ has associated matrix $(AB)^{-1} = B^{-1}A^{-1}$, and this is the matrix we must find. Since the product of diagonal matrices is just the product of diagonal entries, we can easily see that

$$A^{-1} = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -\frac{1}{3} \end{bmatrix}$$

Applying the usual procedure for finding the inverse to B we get

$$\left[\begin{array}{ccc|ccc} 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & -1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & -1 & 0 \end{array}\right]$$

Hence

$$B^{-1} = \begin{bmatrix} -1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & -1 & 0 \end{bmatrix}$$

so

$$B^{-1}A^{-1} = \begin{bmatrix} -1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -\frac{1}{3} \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & -1 & -\frac{1}{3} \\ 0 & -1 & 0 \\ \frac{1}{2} & 1 & 0 \end{bmatrix}$$

5. (8 points) Consider the following 5×5 matrices:

$$M = \begin{bmatrix} 2 & 5 & \sqrt[5]{3} & 2 & \sqrt{2} \\ -3 & 8 & 3 & -6 & 1 \\ \pi & 52 & e & 3 & 5 \\ \sqrt{3} & 2 & 9 & 4 & 7\sqrt{13} \\ 5 & \pi^4 & -1 & 3 & 2 \end{bmatrix} \qquad D = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 & -2 \end{bmatrix}$$

An absolutely horrendous computation shows that M is invertible (You don't need to show this) so you may assume that M^{-1} exists. Define a new matrix $A = MDM^{-1}$. Is A invertible? If so, give a formula for A^{-1} as a product of matrices (You do **not** need to find an explicit formula for M^{-1}). Be sure to carefully explain your reasoning.

To determine if A is invertible, we must compute it's determinant. Recall from class that $\det(A^{-1}) = \frac{1}{det(A)}$. Using this and the fact that $\det(AB) = det(A)det(B)$ we have

$$det(MDM^{-1}) = det(M)det(D)det(M^{-1}) = det(M)det(D)\frac{1}{det(M)} = det(D)$$

Since D is a diagonal matrix, we can easily compute it's determinant, $\det(D) = (2)(-1)(3)(5)(-2) = 60$, hence A is invertible.

To compute it's inverse we apply shoes and socks to obtain

$$(MDM^{-1})^{-1} = (M^{-1})^{-1}D^{-1}M^{-1} = MD^{-1}M^{-1}$$